Weight as the Reading on a Spring Scale—C.E. Mungan, Spring 2001

Most textbooks define weight as the force of gravity which the nearest astronomical body exerts on an object. This has several problems, most of which are associated with the fact that this definition involves an abstract action at a distance, a concept quite difficult for introductory students to grasp. For example, what does one call the "corrected weight" after accounting for the centrifugal force due to Earth's rotation or for the Sun's gravitational attraction? How about the "apparent weight" of an individual aboard an orbiting space shuttle or an accelerating elevator? In light of such issues, an alternative, operational definition of weight as the reading on a spring scale (or equivalently, on a gravimeter) has grown in popularity. In this note, I summarize how this concept would be used in the typical classical mechanics sequence of a physics course.

The root idea appears in the free-fall section of the 1D kinematics chapter and the projectile motion discussion in 2D kinematics. We consider an object which meets the following conditions:

• it is falling freely under the influence of gravity alone—that is, it has no internal propulsion system, air resistance is negligible, it is not being restrained by contact with anything (surfaces, springs, strings, etc.), and no electromagnetic forces are acting upon it;

• it is in the vicinity of Earth's surface (to be specific, say within 10 km);

• it is observed by an individual at rest on Earth (i.e., in Earth's frame of reference).

We call this "surface freefall motion in Earth's frame" or more simply "projectile motion" for brevity. Experiments are now either actually performed or cited to show that the acceleration under these conditions is approximately constant in magnitude and direction, which we write as

$$\mathbf{a}_{proj} = 9.8 \text{ m/s}^2 \text{ downward}. \tag{1}$$

All the usual kinematics problems can now be worked.

Next forces are introduced, defined operationally as pushes or pulls on an object. Given that forces are always local, there is little merit in distinguishing contact and field forces. Newton's first law is a statement about the existence of inertial reference frames. Inertial mass $m_{inertial}$ is introduced as a measure of the inertia of an object, that is, its resistance to accelerations via Newton's second law. A catalog of forces follows, beginning with the normal force. Newton's third law is then introduced and weight is defined as the third-law counterpart to a vertically supporting normal force, i.e., what a bathroom scale would read if the object were placed on it in the local freefall orientation with no other forces acting on it. It is worth emphasizing that the constancy of $\mathbf{a}_{proj}$ in the afore-mentioned experiments implies weight is independent of velocity, i.e., in order to place a moving object on a scale, one is permitted to first stop the object. The physical origin of this weight force is explained to be due to an interaction between an intrinsic property of the object called its gravitational mass m_{grav} and a local quantity called the gravitational field $\mathbf{g}$. The value of this field at a point in space has units of N/kg and depends on two things: the location of that point relative to all other gravitational masses of significance and the choice of the reference frame in which this field is measured. The first effect can be quantified using the universal law of gravitation, which can be introduced at this juncture. Neglecting all large bodies other than the Earth, we obtain

$$\mathbf{g}(r) = -\frac{GM_E}{r^2}\hat{\mathbf{r}}\,. \tag{2}$$

The important issue concerning the second effect is whether the reference frame is inertial, as was implicitly assumed in writing Eq. (2). If it is not, we subscript **g** with the name of the frame, whose acceleration must be known. The following four examples can be analyzed.

First, let's idealize and suppose the Earth to be perfectly spherical, of uniform density, nonrotating, and lacking an atmosphere, with no other astronomical bodies in the vicinity. Now consider an object near its surface and suppose we wish to measure its weight in Earth's frame of reference which is inertial under these idealized conditions. To do so, we simply drop the object so that it falls freely and apply Newton's second law,

$$\mathbf{F} = m_{inertial}\mathbf{a} \quad \Rightarrow \quad m_{grav}\mathbf{g}(R_E) = m_{inertial}\mathbf{a}_{proj}\,. \tag{3}$$

Substituting Eqs. (1) and (2), together with the known values for the mass and radius of Earth and for the universal gravitational constant, we find two remarkable facts,

$$\boxed{\mathbf{a}_{proj} = \mathbf{g}(R_E)} \text{ and } \boxed{m_{grav} = m_{inertial} \equiv m}. \tag{4}$$

The first result accounts for our experimentally determined value of the acceleration of a projectile. Note the unit consistency, $\mathrm{m/s^2 = N/kg}$. In imperial units, $1\ \mathrm{lb} \equiv 4.448\ \mathrm{N} = 2.2^{-1}\ \mathrm{kg} \times 9.8\ \mathrm{m/s^2}$, from which one deduces the familiar fact that 2.2 kilograms weigh one pound at Earth's surface. The second result is called the principle of equivalence—historically minded lecturers might enjoy a brief detour into general relativity to discuss the significance of this "coincidence."

As a second example, consider a small satellite (such as the space shuttle) orbiting around our idealized Earth from the point of view of two different observers. First, suppose an astronaut aboard the shuttle, which is a noninertial frame of reference, observes an apple floating motionlessly with respect to it. The apple is weightless because if the astronaut placed it on a scale the reading would remain zero, i.e.,

$$g_{shuttle}(r_{orbit}) = 0\,. \tag{5}$$

In contrast, consider a second observer standing on a tall tower with its base attached to the Earth and its platform at the instantaneous location of the shuttle. He snatches the apple as it passes by and places it on his own scale, thereby deducing that

$$g(r_{orbit}) = \left(\frac{R_E}{r_{orbit}}\right)^2 9.8\ \mathrm{m/s^2}. \tag{6}$$

The tower observer points out that this nonzero gravitational field is equal to the centripetal acceleration of the orbiting apple; the astronaut, equally correctly, notes that weightlessness is consistent with the fact that the apple remains at rest in his frame of reference. The transformation equation for the gravitational field is

$$\mathbf{g}_{noninertial} = \mathbf{g}_{inertial} - \mathbf{a}_{frame} \tag{7}$$

where $\mathbf{a}_{frame}$ is the acceleration of the noninertial frame as measured by an inertial observer. This is simply the derivative of the Galilean transformation law for relative velocities. Weight is thus frame-dependent and one must deduce the reference frame from the context if it is not explicitly stated.

As an everyday example of the consequences of Eq. (7), consider next a woman of mass m standing on the floor of an elevator accelerating downward with magnitude a. In earth's frame, her weight is $mg(R_E)$, while in her own frame it is $m[g(R_E) - a]$. She weighs less and can directly feel this as a more jaunty spring in her stride if she walks around. This result can also be calculated as the magnitude of the normal force with which the elevator floor pushes upward on her. The limiting case is if the elevator cable breaks so that it falls freely with $a = a_{proj} = g(R_E)$. In that case the woman is weightless: she can freely float around inside along with any other contents of the elevator.

Finally we consider the effects of the real Earth. The rotation results in centrifugal and Coriolis forces. The former results in a deviation away from the purely radial direction and into the polar direction of a plumb bob at acute angles of latitude. The latter is responsible, for example, for the circulation of hurricanes. The sun and moon have ratios of mass to distance squared which are large enough to have measurable gravitational effects on earth, as evidenced by the tides for instance. Modern gravimeters are sensitive enough to detect the asphericity and inhomogeneity of the earth. There is a small equatorial bulge, the effect of which can easily be estimated using Eq. (2). Nonuniformities in density may indicate the presence of heavy mineral deposits or light oil cavities, and hence their detection is a valuable prospecting tool. All of the preceding effects are included in a spring scale measurement; hence Eq. (4) remains valid. Lastly we can account for air resistance by measuring the acceleration of a projectile only at instants in its trajectory when its speed and hence the drag is low. Best of all is to extrapolate the motion of an object dropped from rest back to $t = 0$.